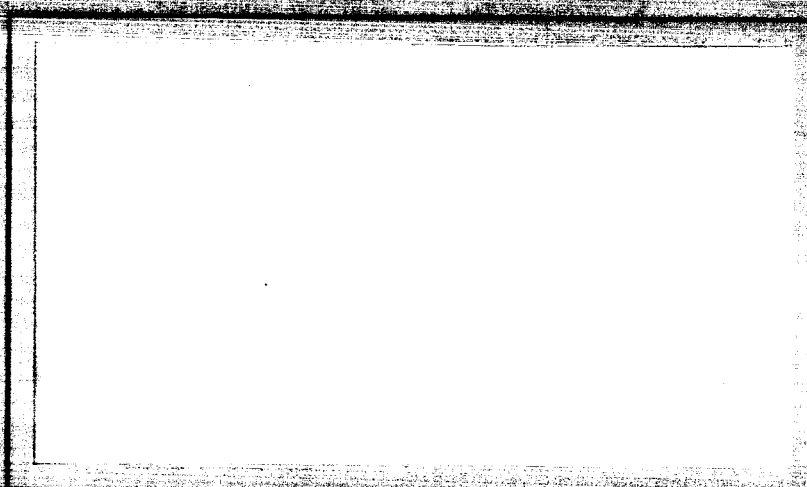


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FOR THE
CALCULUS OF VARIATIONS ★

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SUMMARY

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A differential correction scheme is developed for the improvement of the approximate initial values of the adjoint variables so that an integral functional satisfying desired boundary conditions is optimized. The adjoint variables satisfy a system of equations that are developed by applying the classical methods of the calculus of variations, properly extended, or Pontryagin's maximum principle. Approximate initial values for the adjoint variables are assumed.

A general transition matrix is derived for the variations of the end conditions caused by the variations of the initial values of the adjoint variables, including the variations of the thrusting program and of the final time of the nominal optimum trajectory. An iteration scheme also is discussed for the convergence of the differential corrections to the desired end conditions.

AUTHOR

LIST OF SYMBOLS

a	Semimajor axis of Kepler orbit
c	Gas exhaust velocity
E	Eccentric anomaly
\underline{e}	Unit vector along the thrust direction
$F(t)$	Partials of the vector functions \underline{f} and \underline{g} with respect to the vectors of state variables \underline{x} and adjoint variables \underline{y}
$f_0(\underline{x}, \underline{u})$	Integral functional to be optimized
$\underline{f}(\underline{x}, \underline{u})$	Vector function of state variables (n-dimensional)
$\underline{f}(\underline{x}, \underline{u}, \underline{y})$	General form of vector state variables
f, g, \dot{f}, \dot{g}	Scalar functions relating position and velocity vectors a time t with initial position and velocity vectors for the Kepler problem
$\underline{g}(\underline{x}, \underline{u}, \underline{y})$	General form of vector adjoint variables
$\mathcal{H}(\underline{x}, \underline{u}, \underline{y})$	Hamiltonian
\underline{H}	Angular momentum vector $\underline{R} \times \dot{\underline{R}}$
h	Magnitude of angular momentum
$m(t)$	Mass of space vehicle
N	Number of switchings of thrusting program
n	Mean motion
\underline{R}	Position vector of the vehicle
$\dot{\underline{R}}$	Velocity vector of the vehicle
$P(t)$	Transformation of variations of conventional state variables to those of the orbit parameters

\underline{r}	General vector of state and adjoint variables
r	Magnitude of position vector
$S(t)$	Switching function for engine, "on" or "off"
T	Final time
t	Time
$u(t)$	Control function of time
U	Control region (independent of time)
\underline{V}	Velocity vector of vehicle
v	Magnitude of velocity vector
$X_{x,y}(T, t_0)$	Transition matrix of the partials $\frac{\partial \underline{x}(T)}{\partial \underline{x}(t_0)}$ and $\frac{\partial \underline{x}(T)}{\partial \underline{y}(t_0)}$
$x_0(T)$	Integral to be optimized
$\underline{x}(t)$	State vector variables (n-dimensional)
$\underline{\tilde{x}}(t)$	Augmented state vector (x_0, \underline{x})
$\underline{y}(t)$	Vector of adjoint variables (n-dimensional)
$\underline{\tilde{y}}(t)$	Augmented vector of adjoint variables (y_0, \underline{y})
$Y_{x,y}(T, t_0)$	Transition matrix of the partials $\frac{\partial \underline{y}(T)}{\partial \underline{x}(t_0)}$ and $\frac{\partial \underline{y}(T)}{\partial \underline{y}(t_0)}$ respectively

GREEK LETTERS

$\underline{\alpha}(t)$	Set of orbit parameters
$[\Gamma(T, t_0)]$	General transition matrix of $\frac{\partial \underline{x}(T)}{\partial \underline{y}(t_0)}$ including the optimum change of thrusting program
$[\hat{\Gamma}]$	The first six rows of the general transition matrix $[\Gamma]$
Γ_7	The last row of the general transition matrix $[\Gamma]$
$\delta \dot{\underline{x}}(t_j)$	$\lim_{\epsilon \rightarrow 0} [\dot{\underline{x}}(t_j - \epsilon) - \dot{\underline{x}}(t_j + \epsilon)]$ at time t_j of change of thrusting program
$\delta \dot{\underline{y}}(t_j)$	$\lim_{\epsilon \rightarrow 0} [\dot{\underline{y}}(t_j - \epsilon) - \dot{\underline{y}}(t_j + \epsilon)]$ at time t_j of change of thrusting program
δ_{ij}	Kroneker's delta
$\Delta \underline{\alpha}(t)$	Variation of the set of orbit parameters

$\Delta \underline{h}(t)$	Variation of the general vector of state and adjoint variables due to the control vector change $\Delta \underline{u}$
$\Delta \underline{f}(t)$	Variation of the vector function of the state variables due to control vector change $\Delta \underline{u}$
$\Delta \underline{g}(t)$	Variation of the vector function of the adjoint variables due to the control vector change $\Delta \underline{u}$
$\Delta \underline{r}(t)$	Variation of the general vector of state and adjoint variables
$\Delta S(t)$	Variation of the switching function $S(t)$
ΔT	Variation of the final time T
θ	Eccentric anomaly measured from initial position
$\underline{\lambda}$	Vector of adjoint variables (y_4, y_5, y_6)
μ	Gravitational constant times mass of the attracting body
$\underline{\nu}$	Vector of adjoint variables (y_1, y_2, y_3)
$\Phi(t, t_0)$	Transition matrix relating variations of the state variables \underline{x} and the adjoint variables \underline{y} at time t with those at t_0
$\Psi(t, t_0)$	Transition matrix of the set of orbit parameters
$[\Omega]$	General transition matrix of $\frac{\partial \underline{y}(T)}{\partial \underline{y}(t_0)}$ including the optimum change of the thrusting program
$[\hat{\Omega}]$	The first six rows of the general transition matrix $[\Omega]$
Ω_7	The last row of the general transition matrix $[\Omega]$

SUBSCRIPTS

i, j	Components
o	Initial value of time t_0

SUPERSCRIPTS

\bullet	Differentiation wrt time
T	Transpose of a matrix
\wedge	Vector or matrix reduced to six rows
-1	Inverse of a matrix

INTRODUCTION

In the problems of the calculus of variations, a system of partial differential equations must be solved with specified boundary conditions. In addition to the state and control variables that appear in the equations of motion, the inequalities of constraints, and the functional that should be optimized, there is a number of adjoint variables that satisfy additional equations for the optimization of the given system. These equations are derived by the application of the classical methods of the calculus of variations, properly extended, or from Pontryagin's maximum principle [1], [2].

When some approximate values of the adjoint variables at the initial time t_0 have been calculated, then, by numerical integration of the above systems of equations, an optimal solution is obtained that does not satisfy the desired end conditions. In this paper, a differential correction scheme is developed that will improve the approximate initial values of the adjoint variables so that the optimal solution will satisfy the desired end conditions. A general transition matrix is derived for the variations of the end conditions caused by the variations of the initial values of the adjoint variables, including the variations of the thrusting program of the nominal optimum trajectory and the variation of the final time. An iteration scheme also is presented for the convergence of the improved values of the adjoint variables to those of the optimum solution.

First, the general equations of the state variables, used mostly as constraints, are given, together with the equations of the adjoint variables. Second, the variational equations for the above systems of equations are derived, and an application to the problem of minimizing the fuel of a space vehicle flying between two given boundary points is given as an example. Third, a differential correction scheme is derived for the improvement of the approximate initial values of the adjoint variables, and an iteration scheme is presented for the convergence of the improved values of the adjoint variables, so that the optimum solution will satisfy the desired end conditions. Finally, conclusions and recommendations are presented for the application of this scheme to the actual flight of space vehicles.

FUNDAMENTAL SYSTEM OF EQUATIONS

State Variables

The motion of a vehicle is characterized by the vector variable $\underline{x}(t)$ belonging to the vector space W at any instant of time t . It is assumed that this motion is controlled by a control vector $\underline{u}(t)$.

The fundamental system of equations of state variables is given by

$$\dot{\underline{x}}_i(t) = f_i(\underline{x}(t), \underline{u}(t)) \quad (i = 1, 2, \dots, n) \quad (1)$$

where $\underline{x}(t)$ is an n -dimensional piecewise differentiable state vector, and $\underline{u}(t)$ is an r -dimensional piecewise continuous control vector belonging to an arbitrary control region U that is independent of time. The functions f_i are defined for $\underline{x} \in W$ and for $\underline{u} \in U$ and are assumed to be continuous in the variables $\underline{x}(t)$ and $\underline{u}(t)$ and continuously differentiable with respect to $\underline{x}(t)$. For a certain admissible control $\underline{u}(t)$, the motion of the vehicle $\underline{x}(t)$ is uniquely determined.

The integral functional to be optimized is

$$x_0(T) = \int_{t_0}^T f_0(\underline{x}(t), \underline{u}(t)) dt \quad (2)$$

The necessary conditions for the optimum control vector $\underline{u}(t)$ of Eq.(2) are formulated for fixed boundary conditions of the state variables $\underline{x}(t_0)$ and $\underline{x}(T)$ and for free end time T .

Adjoint Variables

For the optimum solution of Eq. (2), another system of equations is considered. This system is linear and homogeneous in the adjoint variables $\underline{y}(t) = (y_0, y_1, \dots, y_n) = (y_0, \underline{y})$ which is an $(n+1)$ -dimensional continuous vector, and is given by

$$\dot{y}_i(t) = - \sum_{j=0}^n \frac{\partial f_j(\underline{x}(t), \underline{u}(t))}{\partial x_i} y_j(t) \quad (i = 0, 1, \dots, n) \quad (3)$$

The Hamiltonian $\mathcal{H}(\underline{x}(t), \underline{u}(t), \underline{y}(t))$ is defined by

$$\mathcal{H}(\underline{x}, \underline{u}, \underline{y}) = \sum_{i=0}^n y_i(t) f_i(\underline{x}(t), \underline{u}(t)) \quad (4)$$

and the systems of Eqs. (1), (2), and (3) correspond to the Hamiltonian system

$$\begin{aligned} \dot{\underline{x}}_i(t) &= \frac{\partial \mathcal{H}}{\partial y_i} \\ \dot{\underline{y}}_i(t) &= -\frac{\partial \mathcal{H}}{\partial x_i} \end{aligned} \quad (5)$$

Pontryagin's maximum principle and transversality condition give, for optimal $\underline{x}_0(T)$, the function $\mathcal{H}(\underline{x}(t), \underline{u}(t), \underline{y}(t))$ of $\underline{u}(t)$ belonging to U attains its maximum at the point $\underline{u}(t)$, i.e.

$$\mathcal{H}(\underline{x}(t), \underline{u}(t), \underline{y}(t)) = \sup_{\underline{u} \in U} \mathcal{H}(\underline{x}(t), \underline{u}(t), \underline{y}(t)) = 0 \quad (6)$$

$$y_0(t_0) \leq 0 \quad \text{and} \quad y_k(T) = 0$$

where the subscript k corresponds to the subscript of the state variables for which the terminal value $x_k(T)$ is free. For most of the engineering applications, we have $y_0 \neq 0$, which is normalized to $y_0 = -1$.

The Lagrangian multipliers $\lambda^{(L)}(t)$ of the classical calculus of variations are related to the adjoint variables $\underline{y}(t)$ by the relationship

$$\lambda_{i \neq 0}^{(L)}(t) = \frac{\partial f_0(\underline{x}(t), \dot{\underline{x}}(t), \underline{u}(t))}{\partial \dot{x}_i} y_0(t) + y_i(t) \quad (7)$$

If the time t appears explicitly in the system of functions f or f_0 , then it always can be transformed to an autonomous system by introducing an auxiliary state variable that is defined by

$$\dot{x}_{n+1}(t_0) = 1 \quad \text{with} \quad x_{n+1}(t_0) = t_0 \quad (8)$$

Example

For a space vehicle powered by a throttled engine and flying in the gravitational field of only one attracting body, the system of equations of the state variables, i.e., Eq. (1), reduces to

$$\begin{aligned}\dot{\underline{R}} &= \underline{V} & f_1, f_2, f_3 \\ \dot{\underline{V}} &= -\frac{\mu}{r^3} \underline{R} + \frac{u(t)}{m} \underline{e} & f_4, f_5, f_6 \\ \dot{m} &= -\frac{u(t)}{c} & f_7\end{aligned}\quad (9)$$

where \underline{e} is a unit vector in the direction of the thrust, and $u(t)$ is the control variable belonging to the range $0 \leq u(t) \leq K$.

For minimizing the fuel between $\underline{x}(t_0)$ and $\underline{x}(T)$ with free end time, the integral functional to be optimized, i.e., Eq. (2), becomes

$$x_0(T) = \int_{t_0}^T f_0(\underline{x}(t), u(t)) dt \quad (10)$$

$$\text{with } f_0 = -\dot{m} = \frac{u(t)}{c}.$$

The system of the adjoint variables, i.e., Eq. (3), reduces to

$$\begin{aligned}\dot{y}_0(t) &= 0 \\ \dot{\underline{y}}(t) &= \frac{\mu}{r^3} \underline{\lambda} - 3\mu \frac{\underline{R} \cdot \underline{\lambda}}{r^5} \underline{R} \\ \dot{\underline{\lambda}}(t) &= -\underline{\nu} \\ \dot{y}_7(t) &= \frac{u(t)}{m^2} (\underline{\lambda} \cdot \underline{e})\end{aligned}\quad \begin{aligned}\underline{\nu} &= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\ \underline{\lambda} &= \begin{bmatrix} y_4 \\ y_5 \\ y_6 \end{bmatrix}\end{aligned}\quad (11)$$

The maximum principle and the transversality conditions of Eq. (6) become

$$\begin{aligned} \mathcal{H} = \sup_{u \in U} \mathcal{H} &= y_0 f_0 + \underline{v} \cdot \underline{V} + \underline{\lambda} \cdot \left(\frac{\mu}{r^3} \underline{R} + \frac{u(t)}{m} \underline{e} \right) - y_7 \frac{u(t)}{c} = 0 \\ y_0(t) &= -1 \quad \text{and} \quad y_7(T) = 0 \end{aligned} \quad (12)$$

where $f_0 = \frac{u(t)}{c}$.

From Eq. (1), it is obvious that $\underline{\lambda} // \underline{e}$ and that the switching function for $u = 0$ or $u = K$ is defined by

$$S(t) = \frac{|\underline{\lambda}|}{m} - \frac{y_7 - y_0}{c} \gtrless 0 \quad (13)$$

when $u(t) = \begin{cases} K & (\text{max}) \\ 0 & (\text{min}) \end{cases}$ respectively.

VARIATIONAL EQUATIONS

In this section, the variational equations of the optimum trajectory of a space vehicle are derived. The formulation of these equations is required for the application of the differential correction scheme that is developed in the next section.

The application of Pontryagin's maximum principle for the solution of optimal problems yields additional information for the synthesis of optimal controls. Making use of this principle, the system of Eqs. (1) and (3) may be rewritten in the following general form.

$$\dot{\underline{r}}(t) \equiv \begin{bmatrix} \dot{\underline{x}}(t) \\ \dot{\underline{y}}(t) \end{bmatrix} = \begin{bmatrix} \underline{f}(\underline{x}, \underline{y}, \underline{u}) \\ \underline{g}(\underline{x}, \underline{y}, \underline{u}) \end{bmatrix} \quad (14)$$

The variations of this system are obtained by

$$\Delta \dot{\underline{r}}(t) = F(t) \Delta \underline{r}(t) + \Delta \underline{h}(t) \quad (15)$$

where the matrix $F(t)$ and the vector $\Delta \underline{h}(t)$ are given by

$$F(t) = \begin{bmatrix} \frac{\partial \underline{f}}{\partial \underline{x}} & \frac{\partial \underline{f}}{\partial \underline{y}} \\ \frac{\partial \underline{g}}{\partial \underline{x}} & \frac{\partial \underline{g}}{\partial \underline{y}} \end{bmatrix} \quad (16)$$

$$\Delta \underline{h}(t) = \begin{bmatrix} \Delta \underline{f} \\ \Delta \underline{g} \end{bmatrix} = \begin{bmatrix} \underline{f}(\underline{u} + \Delta \underline{u}) - \underline{f}(\underline{u}) \\ \underline{g}(\underline{u} + \Delta \underline{u}) - \underline{g}(\underline{u}) \end{bmatrix}$$

Transition Matrix

The fundamental solution matrix for the homogeneous part of Eq. (15), i.e.,

$$\dot{\Phi}(t) = F(t) \Phi(t)$$

with initial conditions $\Phi(t_0, t_0) = I$ (unit matrix), is the transition matrix $\Phi(t, t_0)$ of the system. From the properties of the fundamental solution matrix and the transition matrix $\Phi(t, t_0)$, we obtain

$$\Delta \underline{r}(t) = \Phi(t, t_0) \Delta \underline{r}(t_0) + \int_{t_0}^t \Phi(t, \tau) \Delta \underline{h}(\tau) d\tau \quad (17)$$

which is the solution of the non-homogeneous Eq. (15).

In the example of the powered space vehicle flying in the gravitational field of one attracting body, Eq. (17) reduces to

$$\Delta \underline{r}(T) = \Phi(T, t_0) \Delta \underline{r}(t_0) + \sum_{j=1}^N \Phi(T, t_j) \Delta \underline{h}(t_j) \Delta t_j \quad (18)$$

where t_j is the time at which the thrusting program of the optimum nominal trajectory with the approximate values of initial conditions $\underline{r}(t_0)$ switches "on" or "off" during the time interval $t_0 < t_j < T$, and $\Delta \underline{r}(T)$ gives the deviations of the nominal end conditions from the desired end conditions, i.e.

$$\Delta \underline{r}(T) = \begin{bmatrix} \Delta \underline{x}(T) \\ \Delta \underline{y}(T) \end{bmatrix}$$

$$\Phi(T, t_0) = \begin{bmatrix} \frac{\partial \underline{x}(T)}{\partial \underline{x}(t_0)} & \frac{\partial \underline{x}(T)}{\partial \underline{y}(t_0)} \\ \frac{\partial \underline{y}(T)}{\partial \underline{x}(t_0)} & \frac{\partial \underline{y}(T)}{\partial \underline{y}(t_0)} \end{bmatrix} = \begin{bmatrix} X_x(T, t_0) & X_y(T, t_0) \\ Y_x(T, t_0) & Y_y(T, t_0) \end{bmatrix} \quad (19)$$

$$\Delta h(t_j) = \lim_{\epsilon \rightarrow 0} \begin{bmatrix} \underline{\dot{x}}(t_j - \epsilon) - \underline{\dot{x}}(t_j + \epsilon) \\ \underline{\dot{y}}(t_j - \epsilon) - \underline{\dot{y}}(t_j + \epsilon) \end{bmatrix} = - \begin{bmatrix} \delta \underline{\dot{x}}(t_j) \\ \delta \underline{\dot{y}}(t_j) \end{bmatrix}$$

Because the boundary conditions of the state variables at the initial time t_0 are given, we have $\Delta \underline{x}(t_0) \equiv 0$, and Eq. (18) becomes (see Fig. 1)

$$\Delta \underline{r}(T) = \Phi(T, t_0) \Delta \underline{r}(t_0) - \sum_{j=1}^N \Phi(T, t_j) \delta \underline{\dot{r}}(t_j) \Delta t_j \quad (20)$$

or

$$\begin{bmatrix} \Delta \underline{x}(T) \\ \Delta \underline{y}(T) \end{bmatrix} = \begin{bmatrix} X_x & X_y \\ Y_x & Y_y \end{bmatrix} \begin{bmatrix} 0 \\ \Delta \underline{y}(t_0) \end{bmatrix} - \sum_{j=1}^N \begin{bmatrix} X_x^{(j)} & X_y^{(j)} \\ Y_x^{(j)} & Y_y^{(j)} \end{bmatrix} \begin{bmatrix} \delta \underline{\dot{x}}(t_j) \\ \delta \underline{\dot{y}}(t_j) \end{bmatrix} \Delta t_j \quad (21)$$

where $X = X(T, t_0)$, and $X^{(j)} = X(T, t_j)$.

From Eq. (21), we get

$$\Delta \underline{x}(T) = \underline{X}_y \Delta \underline{y}(t_0) - \sum_{j=1}^N \left[\underline{X}_x^{(j)} \delta \underline{\dot{x}}(t_j) + \underline{X}_y^{(j)} \delta \underline{\dot{y}}(t_j) \right] \Delta t_j \quad (22)$$

and

$$\Delta \underline{y}(T) = \underline{Y}_y \Delta \underline{y}(t_0) - \sum_{j=1}^N \left[\underline{Y}_x^{(j)} \delta \underline{\dot{x}}(t_j) + \underline{Y}_y^{(j)} \delta \underline{\dot{y}}(t_j) \right] \Delta t_j \quad (23)$$

Thrusting Program

In the formulation of the variational equations of the optimum nominal trajectory, the time variation Δt_j of the optimum thrusting program has been included where t_j is the time at which the thrust switches "on" or "off" and the switching function of the nominal trajectory is zero, i.e., $S(t_j) = 0$. The time variation Δt_j is calculated from the variation of the switching function $\Delta S(t_j + \Delta t_j)$ for which

$$S(t_j + \Delta t_j) + \Delta S(t_j + \Delta t_j) = 0 \quad (24)$$

From the linear expansion of Eq. (24) we get

$$\dot{S}(t_j) \Delta t_j \approx - \frac{\partial S}{\partial \underline{r}} \Delta \underline{r}(t_j + \Delta t_j) \quad (25)$$

Because $\Delta \underline{r}(t_j + \Delta t_j) \approx \Delta \underline{r}(t_j) + \Delta \dot{\underline{r}}(t_j) \Delta t_j$ and $\frac{\partial S}{\partial \underline{r}} \Delta \dot{\underline{r}}(t_j) = 0$, Eq. (25) becomes

$$\dot{S}(t_j) \Delta t_j \approx - \frac{\partial S}{\partial \underline{r}} \Delta \underline{r}(t_j) \quad (26)$$

Expanding the variation $\Delta \underline{r}(t_j)$ from Eq. (20), we get

$$\Delta \underline{r}(t_j) = \Phi(t_j, t_0) \Delta \underline{r}(t_0) - \sum_{i=1}^{j-1} \Phi(t_j, t_i) \delta \underline{\dot{r}}(t_i) \Delta t_i \quad (27)$$

$$\Delta t_j = \frac{-1}{\dot{S}(t_j)} \frac{\partial S(t_j)}{\partial \underline{r}(t_j)} \left[\Phi(t_j, t_0) \Delta \underline{r}(t_0) - \sum_{i=1}^{j-1} \Phi(t_j, t_i) \delta \underline{\dot{r}}(t_i) \Delta t_i \right] \quad (28)$$

and, in terms of the variations $\Delta \underline{y}(t_0)$, it becomes

$$\begin{aligned} \Delta t_j = & -\frac{1}{\dot{S}(t_j)} \left[\frac{\partial S(t_j)}{\partial \underline{x}(t_j)} X_y(t_j, t_0) + \frac{\partial \dot{S}(t_j)}{\partial \underline{y}(t_j)} Y_y(t_j, t_0) \right] \Delta \underline{y}(t_0) \\ & + \frac{1}{\dot{S}(t_j)} \frac{\partial S(t_j)}{\partial \underline{x}(t_j)} \sum_{i=1}^{j-1} \left[X_x(t_j, t_i) \delta \underline{\dot{x}}(t_i) + X_y(t_j, t_i) \delta \underline{\dot{y}}(t_i) \right] \Delta t_i \\ & + \frac{1}{\dot{S}(t_j)} \frac{\partial S(t_j)}{\partial \underline{y}(t_j)} \sum_{i=1}^{j-1} \left[Y_x(t_j, t_i) \delta \underline{\dot{x}}(t_i) + Y_y(t_j, t_i) \delta \underline{\dot{y}}(t_i) \right] \Delta t_i \end{aligned} \quad (29)$$

From Eq. (13) for the switching function $S(t)$, we find that

$$\begin{aligned} S(t) &= \frac{|\lambda|}{m} - \frac{y_7 - y_0}{c} & \dot{S}(t) &= \frac{\lambda \cdot \dot{\lambda}}{m |\lambda|} \\ \frac{\partial S(t_j)}{\partial \underline{x}(t_j)} &= \left\{ 0, 0, 0, 0, 0, 0, -\frac{|\lambda|}{m^2} \right\} \\ \frac{\partial S(t_j)}{\partial \underline{y}(t_j)} &= \left\{ \frac{y_4}{m |\lambda|}, \frac{y_5}{m |\lambda|}, \frac{y_6}{m |\lambda|}, 0, 0, 0, -\frac{1}{c} \right\} \end{aligned} \quad (30)$$

DIFFERENTIAL CORRECTION SCHEME

Correction Scheme

In this section, a differential correction scheme is developed for the improvement of the approximate initial values of the adjoint variables so that the optimum solution of the problem can be found. The variations of the nominal optimum trajectory of the space vehicle, calculated for the approximate initial values of the adjoint variables, have been derived previously.

Making use of Eqs. (17), we solve for $\Delta \underline{x}(t_0)$ if we know the variation $\Delta \underline{x}(T)$ at the terminal time T . In the example of the powered space vehicle we derived Eqs. (22) and (23) for the variations of $\Delta \underline{x}(T)$ and $\Delta \underline{y}(T)$ caused by the variations of the adjoint variables $\Delta \underline{y}(t_0)$ at the initial time t_0 and the variations Δt_j at the time t_j of the thrusting program, which corresponds to the optimum nominal trajectory for the approximate adjoint variables.

Free End Time

In the case of free end time T , a variation in the terminal time also is taken into consideration, and, making use of Eqs. (29), we find that

$$\Delta \underline{x}(T) = [\Gamma] \Delta \underline{y}(t_0) + \dot{\underline{x}}(T) \Delta T \quad (31)$$

$$\Delta \underline{y}(T) = [\Omega] \Delta \underline{y}(t_0) + \dot{\underline{y}}(T) \Delta T \quad (32)$$

Separating the seventh row of Eqs. (31) and (32), we get

$$\Delta \hat{\underline{x}}(T) = [\hat{\Gamma}] \Delta \underline{y}(t_0) + \hat{\dot{\underline{x}}}(T) \Delta T \quad (33)$$

$$\Delta y_7(T) = \Omega_7 \Delta \underline{y}(t_0) + \dot{y}_7(T) \Delta T \quad (34)$$

where Eqs. (33) and (34) are of the form

$$[6 \times 1] = [6 \times 7] [7 \times 1] + [6 \times 1] [1 \times 1]$$

$$[1 \times 1] = [1 \times 7] [7 \times 1] + [1 \times 1] [1 \times 1]$$

respectively, $[\hat{\Gamma}]$ represents the first six rows of $[\Gamma]$, and Ω_7 represents the seventh row of $[\Omega]$.

For the solution of the system of Eqs. (33) and (34) for $\Delta \underline{y}(t_0)$ and ΔT from the deviations $\Delta \hat{\underline{x}}(T)$ and $\Delta y_7(T) = 0$, we need one more relationship, and this is obtained from Eq. (12), i. e.

$$\mathcal{H}(\underline{x}, \underline{u}, \underline{y}) = \sum_{j=1}^7 y_j \cdot f_j(t) - f_0(t) = 0 \quad (35)$$

Taking the variation of $\mathcal{H}(t)$ at time t_0 , we get

$$\sum_{j=1}^7 f_j(t_0) \Delta y_j(t_0) + \sum_{j=1}^7 y_j(t_0) \Delta f_j(t_0) - \Delta f_0(t_0) = 0 \quad (36)$$

Because $\Delta f_j(t_0) = 0$ and $\Delta f_0(t_0) = 0$ if the variation of the switching function $\Delta S(t_0)$ does not change the sign of $S(t_0)$, Eq. (36) becomes

$$\sum_{j=1}^7 f_j(t_0) \Delta y_j(t_0) = 0 \quad (37)$$

or

$$\underline{v}(t_0) \cdot \Delta \underline{v}(t_0) + \ddot{\underline{R}}(t_0) \cdot \Delta \underline{\lambda}(t_0) - \frac{u(t_0)}{c} \Delta y_7(t_0) = 0 \quad (38)$$

Thus, combining Eqs. (33), (34), and (38), we get eight equations with eight unknown variations that are given by

$$\begin{bmatrix} \Delta \hat{\underline{x}}(T) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} [\hat{\Gamma}] & \hat{\underline{x}}(T) \\ \Omega_7 & \dot{y}_7(T) \\ \dot{\underline{x}}(t_0)^T & 0 \end{bmatrix} \begin{bmatrix} \Delta \underline{y}(t_0) \\ \Delta T \end{bmatrix} \quad (39)$$

Solving for $\Delta \underline{y}(t_0)$ and ΔT , we find that

$$\begin{bmatrix} \Delta \underline{y}(t_0) \\ \Delta T \end{bmatrix} = \begin{bmatrix} [\hat{\Gamma}] & \hat{\underline{x}}(T) \\ \Omega_7 & \dot{y}_7(T) \\ \dot{\underline{x}}(t_0)^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \Delta \hat{\underline{x}}(T) \\ 0 \\ 0 \end{bmatrix} \quad (40)$$

Iteration Scheme

For the calculation of the optimum trajectory of a space vehicle, the differential correction scheme described in this section is applied, and the variation of the adjoint vector $\Delta \underline{y}(t_0)$ at the initial time t_0 , as well as the varia-

tion of the final time ΔT , are derived to match the desired conditions at the final time T in space. Making use of the corrected adjoint variables $\underline{y}_1(t_0) = \underline{y}(t_0) + \Delta \underline{y}(t_0)$, a new optimum nominal trajectory is computed by integrating the system of equations of the state and adjoint variables, i.e., Eqs. (9) and (11), by making use of Eq. (13) for the optimum thrusting program as described previously. Because the differential correction scheme has been derived for linear variations of highly nonlinear equations, it is expected that there still will be a discrepancy between the desired and the new computed values of the end conditions $\Delta \underline{x}_1(T_1)$, where $T_1 = T + \Delta T$.

In general, successive iterations generate corrections $\Delta \underline{y}_k(t_0)$ to the adjoint variables at time t_0 from $\Delta \underline{x}_k(T_k)$ such that

$$\underline{y}_{k+1}(t_0) = \underline{y}_k(t_0) + \Delta \underline{y}_k(t_0) = \underline{y}(t_0) + \sum_{i=0}^k \Delta \underline{y}_i(t_0) \quad (41)$$

which, in turn, gives end conditions with deviations $\Delta \underline{x}_{k+1}(T_{k+1})$ from their desired values, and

$$T_{k+1} = T + \sum_{i=0}^k \Delta T_i \quad (42)$$

This iteration scheme converges to the desired end conditions of the state vector, provided that the deviations are within the linear range. Departure from the linear range will be indicated when the deviations of the computed nominal end conditions from the desired end conditions $\Delta \underline{x}_1(T_1)$ are comparable to or exceed the deviations $\Delta \underline{x}(T)$. In this case, each step of the iteration scheme described above contains a sub-iteration carried out on a parameter γ_k introduced as a factor multiplying the deviations $\Delta \underline{x}_k(T_k)$. Thus

$$\Delta \underline{x}_k^*(T_k) = \gamma_k \Delta \underline{x}_k(T_k) \quad (43)$$

From $\Delta \underline{x}_k^*(T_k)$, we obtain the correction $\Delta \underline{y}_k^*(t_0)$, which is added to $\underline{y}_k^*(t_0)$ for the k^{th} estimate of the adjoint variables at time t_0 . The sub-iteration consists of the determination of a value of γ_k ($0 < \gamma_k \leq 1$) such that the deviations $\Delta \underline{x}_{k+1}(T_{k+1})$ computed from the corrected adjoint variables, i.e.

$$y_{k+1}(t_0) = y_k(t_0) + \Delta y_k^*(t_0) = y(t_0) + \sum_{i=0}^k \Delta y_i^*(t_0) \quad (44)$$

are comparable to or less than the deviations $\Delta x_k(T_k)$. This procedure is continued until the linear range is reached for which $\gamma_k = 1$ and the iteration scheme converges to the desired end conditions.

It should be noted that the same procedure is followed when parameters other than the state variables are specified as end conditions. Of course, these parameters must be expressible as functions of the state variables.

CONCLUSIONS AND RECOMMENDATIONS

A differential correction has been developed for the improvement of the approximate values of the adjoint variables so that the optimal solution of the problems of the calculus of variations is obtained. The mathematical analysis for the differential correction scheme for the optimum trajectory of a space vehicle with minimum fuel consumption between fixed boundary conditions has been presented. The method developed relies on the variations of the nominal optimum trajectory of the space vehicle calculated for the approximate initial values of the adjoint variables, which are assumed to be given. Techniques for the calculation of these approximate values are not considered in this report.

A general transition matrix has been derived for the variations of the end conditions caused by the variations of the initial values of the adjoint variables, including the variations of the thrusting program of the nominal optimum trajectory and the variation of the final time. An iteration scheme also has been discussed for the convergence of the improved values of the adjoint variables to those of the optimum problem satisfying the desired end conditions. In addition, a method for the case of variations beyond the linear range has been outlined.

This program will be highly useful for the determination of optimum space missions and for optimum orbit transfer for intercept and rendezvous of space

vehicles as well as for optimum navigation and guidance of a space vehicle. Further work in this area is readily suggested. First, techniques should be developed for the approximate initial values of the adjoint variables that are used for the optimum nominal trajectory. Second, this correction scheme could be extended readily to optimum problems with more general types of end conditions than those considered in this report. Finally, a more general differential correction scheme is required for the optimum pursuit of a powered spacecraft, which would involve a statistical-control scheme for the probability law of a randomly moving point.

APPENDIX

VARIATIONAL PARAMETERS

For the calculation of variations of the optimum space trajectories, there is a general matrix introduced that relates the variations of the state and adjoint variables at time t to those at time t_0 . This matrix, called the general transition matrix, requires the computation of the partial derivatives of the state and adjoint variables at two different times, i.e., t_0 and T , and relates their linear variations at these times, including the optimum changes of the thrusting program.

When the thrust is "off," the system of equations for the adjoint variables is "adjoint" to the system of equations for the variations of the state variables, which, in this case, is homogeneous, and the transition matrix of the state variables is used for the calculations of the adjoint variables during the coasting intervals of time, i.e., $t_i < t < t_{i+1}$. In this case, the transition matrix of the state variables $\hat{X}(t_{i+1}, t_i)$ is found from the corresponding Kepler problem, and it is expressed in closed form from the solution of this problem.

The variations of the state variables and the values of the adjoint variables for the coasting interval are given by [3].

$$\begin{aligned}\Delta \hat{\underline{x}}(t_{i+1}) &= \hat{\underline{X}}(t_{i+1}, t_i) \Delta \hat{\underline{x}}(t_i) \\ \hat{\underline{y}}(t_{i+1}) &= [\hat{\underline{X}}^T(t_{i+1}, t_i)]^{-1} \hat{\underline{y}}(t_i)\end{aligned}\tag{45}$$

where

$$\begin{aligned}\hat{\underline{x}}(t)^T &= (x_1, x_2, x_3, x_4, x_5, x_6) \\ \hat{\underline{y}}(t)^T &= (y_1, y_2, y_3, y_4, y_5, y_6)\end{aligned}\tag{46}$$

and

$$\hat{\underline{X}}(t_{i+1}, t_i) = \frac{\partial \hat{\underline{x}}(t_{i+1})}{\partial \hat{\underline{x}}(t_i)}\tag{47}$$

The use of the conventional state variables $\hat{\underline{x}}(t)$, which are position and velocity vectors \underline{R} and $\dot{\underline{R}}$ in cartesian coordinates, has the disadvantage that all of their elements have secular terms that vary rapidly with time. If, instead of the conventional state variables, other parameters are used as state variables, the resultant matrix might be simplified considerably. For example, consider the following parameters and their variations:

$\Delta \alpha_1$	Rotation of \underline{R} about $\dot{\underline{R}}$
$\Delta \alpha_2$	Rotation of $\dot{\underline{R}}$ about \underline{R}
$\Delta \alpha_3$	Rotation of both \underline{R} and $\dot{\underline{R}}$ about \underline{H}
$\Delta \alpha_4$	Change in $\cos(\underline{R}, \dot{\underline{R}})$, keeping v and \underline{R} constant
$\Delta \alpha_5$	Relative change in the semimajor axis $\Delta a/a$, keeping \underline{R} and $\dot{\underline{R}}/v$ constant
$\Delta \alpha_6$	Relative change in the magnitude of the position vector $(\Delta r/r)$, keeping \underline{R}/r and $\dot{\underline{R}}/v$ constant.

The transition matrix corresponding to the above parameters, i.e.

$$\Delta \underline{\alpha}(t) = \Psi(t, t_0) \Delta \underline{\alpha}(t_0)\tag{48}$$

$$\Psi(t, t_0) = \begin{bmatrix} \frac{fv}{v_0} & -\frac{gv}{r_0} & 0 & 0 & 0 & 0 \\ -\frac{\dot{f}r}{v_0} & \frac{\dot{g}r}{r_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{\partial \alpha_3}{\partial \alpha_{40}} & \frac{\partial \alpha_3}{\partial \alpha_{50}} & \frac{\partial \alpha_3}{\partial \alpha_{60}} \\ 0 & 0 & 0 & \frac{\partial \alpha_4}{\partial \alpha_{40}} & \frac{\partial \alpha_4}{\partial \alpha_{50}} & \frac{\partial \alpha_4}{\partial \alpha_{60}} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{r_0 v_0}{r^2} g & \frac{\partial \alpha_6}{\partial \alpha_{50}} & \frac{\partial \alpha_6}{\partial \alpha_{60}} \end{bmatrix} \quad (49)$$

where some of the non-zero elements are listed as partials of the orbital parameters and are given by Ref. [4] as

$$\frac{\partial \alpha_3}{\partial \alpha_{40}} = -\frac{r_0 v_0}{r^2} g \left[\frac{r_0 v_0}{h} \left(\frac{v_0 g}{r_0} + f \alpha_{40} \right) + \frac{rh}{\mu g} (f-1)(\dot{g}-1) \right]$$

$$\frac{\partial \alpha_3}{\partial \alpha_{50}} = \frac{h}{r^2} \left[\frac{\mu}{v_0^2 r_0} g - \frac{3}{2} (t-t_0) - 3f(g-(t-t_0)) + (f-1) \frac{r_0}{v_0} \left(\alpha_{40} - \frac{rv}{r_0 v_0} \alpha_4 \right) \right] \quad (51)$$

$$\frac{\partial \alpha_3}{\partial \alpha_{60}} = \frac{h}{r^2} \left[fg \left(1 - \frac{\mu}{r_0 v_0^2} \right) - 2g + (f-1)^2 \frac{r_0}{v_0} \alpha_{40} \right] \quad (52)$$

$$\frac{\partial \alpha_4}{\partial \alpha_{40}} = \frac{r_0 v_0}{rv} \left[\dot{g} - \frac{\mu \alpha_4}{r^2 v} \left(1 - \frac{r}{a} \right) g \right] \quad (53)$$

$$\frac{\partial \alpha_4}{\partial \alpha_{50}} = \frac{\mu}{rv^2} \left[\left(1 - \frac{r}{a} \right) \left\{ \alpha_4 \left(f \frac{r_0}{v_0} \alpha_{40} + \dot{g} \right) - \frac{3v(t-t_0)}{2r} (1 - \alpha_4^2) \right\} - \frac{v}{v_0} \left(1 - \frac{r_0}{a} \right) \alpha_{40} \right. \\ \left. + \frac{v}{r} \left\{ 1 - \left(1 - \frac{r}{a} \right) \alpha_4^2 \right\} \left\{ \frac{\mu}{v_0^2 r_0} g - \frac{r_0}{v_0} \left(\alpha_{40} - \frac{rv}{r_0 v_0} \alpha_4 \right) \right\} \right] \quad (54)$$

$$\frac{\partial \alpha_4}{\partial \alpha_{60}} = \frac{\mu}{rv^2} \left[\frac{v}{v_0} \left(1 - \frac{r_0}{a} \right) \alpha_{40} - \frac{v}{r} \left\{ 1 - \alpha_4^2 \left(1 - \frac{r}{a} \right) \right\} \left\{ g + \frac{r_0}{v_0} (f-1) \alpha_{40} \right\} \right. \\ \left. - \frac{\mu}{v_0^2 r_0} \left(1 - \frac{r}{a} \right) \left\{ \dot{g} + \frac{r_0}{r} \left(1 - \frac{r}{a} \right) \right\} \right] \quad (55)$$

$$\frac{\partial \alpha_6}{\partial \alpha_{50}} = 1 - \frac{r_0}{v_0} \dot{f} \alpha_{40} - \dot{g} + \frac{v}{r} \alpha_4 \left[\frac{\mu}{v_0^2 r_0} g - \frac{r_0}{v_0} \left(\alpha_{40} - \frac{rv}{r_0 v_0} \alpha_4 \right) - \frac{3}{2} (t-t_0) \right] \quad (56)$$

$$\frac{\partial \alpha_6}{\partial \alpha_{60}} = \frac{\mu}{v_0^2 r_0} \left[\dot{g} + \frac{r_0}{r} \left(1 - \frac{r}{a} \right) \right] - \frac{v \alpha_4}{r} \left[g + \frac{r_0}{v_0} (f-1) \alpha_{40} \right] \quad (57)$$

The transformation relating the variation of the conventional state variable $\Delta \hat{\underline{x}}^T = (\Delta \underline{R}, \Delta \dot{\underline{R}})$ to the variations of the above set of parameters $\Delta \underline{\alpha}^T = (\Delta \alpha_1, \Delta \alpha_2, \dots, \Delta \alpha_6)$ is given by

$$\Delta \hat{\underline{x}}(t) = P(t) \Delta \underline{\alpha}(t) \quad \text{and} \quad \Delta \underline{\alpha}(t) = P(t)^{-1} \Delta \hat{\underline{x}}(t) \quad (58)$$

where

$$P(t) = \begin{bmatrix} \frac{-\underline{H}}{v} & 0 & \frac{\underline{H} \times \underline{R}}{h} & 0 & 0 & \underline{R} \\ 0 & \frac{\underline{H}}{r} & \frac{\underline{H} \times \dot{\underline{R}}}{h} & \frac{-rv}{R^2} \underline{H} \times \dot{\underline{R}} & \frac{\mu}{2v^2 a} \dot{\underline{R}} & \frac{-\mu}{rv^2} \dot{\underline{R}} \end{bmatrix} \quad (59)$$

and

$$P(t)^{-1T} = \begin{bmatrix} \frac{-v\underline{H}}{h^2} & 0 & \frac{\underline{H} \times \underline{R}}{hr^2} & \frac{\underline{H} \times \underline{R}}{r^3 v} & \frac{2a}{r^3} \underline{R} & \frac{\underline{R}}{r^2} \\ 0 & \frac{r\underline{H}}{h^2} & 0 & \frac{-\underline{H} \times \dot{\underline{R}}}{rv^3} & \frac{2a}{\mu} \dot{\underline{R}} & 0 \end{bmatrix} \quad (60)$$

The relationship between the transition matrix $\hat{X}(t, t_0)$ for the conventional state variables $\hat{x}(t)$ and $\Psi(t, t_0)$ for the above set of parameters $\alpha(t)$ is given by

$$\hat{X}(t, t_0) = P(t) \Psi(t, t_0) P(t_0)^{-1} \text{ and } \Psi(t, t_0) = P(t)^{-1} \hat{X}(t, t_0) P(t_0) \quad (61)$$

The scalar functions f, g, \dot{f} , and \dot{g} are given by

(Elliptic)	(Hyperbolic)	
$f = \frac{a}{r_0} (\cos \theta - 1) + 1$	$f = \frac{a}{r_0} (\cosh \theta - 1) + 1$	
$g = (t - t_0) - \frac{\theta - \sin \theta}{n}$	$g = (t - t_0) - \frac{\sinh \theta - \theta}{n}$	(62)
$\dot{f} = \frac{-a^2 n}{r r_0} \sin \theta$	$\dot{f} = -\frac{\sqrt{-\mu a}}{r r_0} \sinh \theta$	
$\dot{g} = \frac{a}{r} (\cos \theta - 1) + 1$	$\dot{g} = \frac{a}{r} (\cosh \theta - 1) + 1$	

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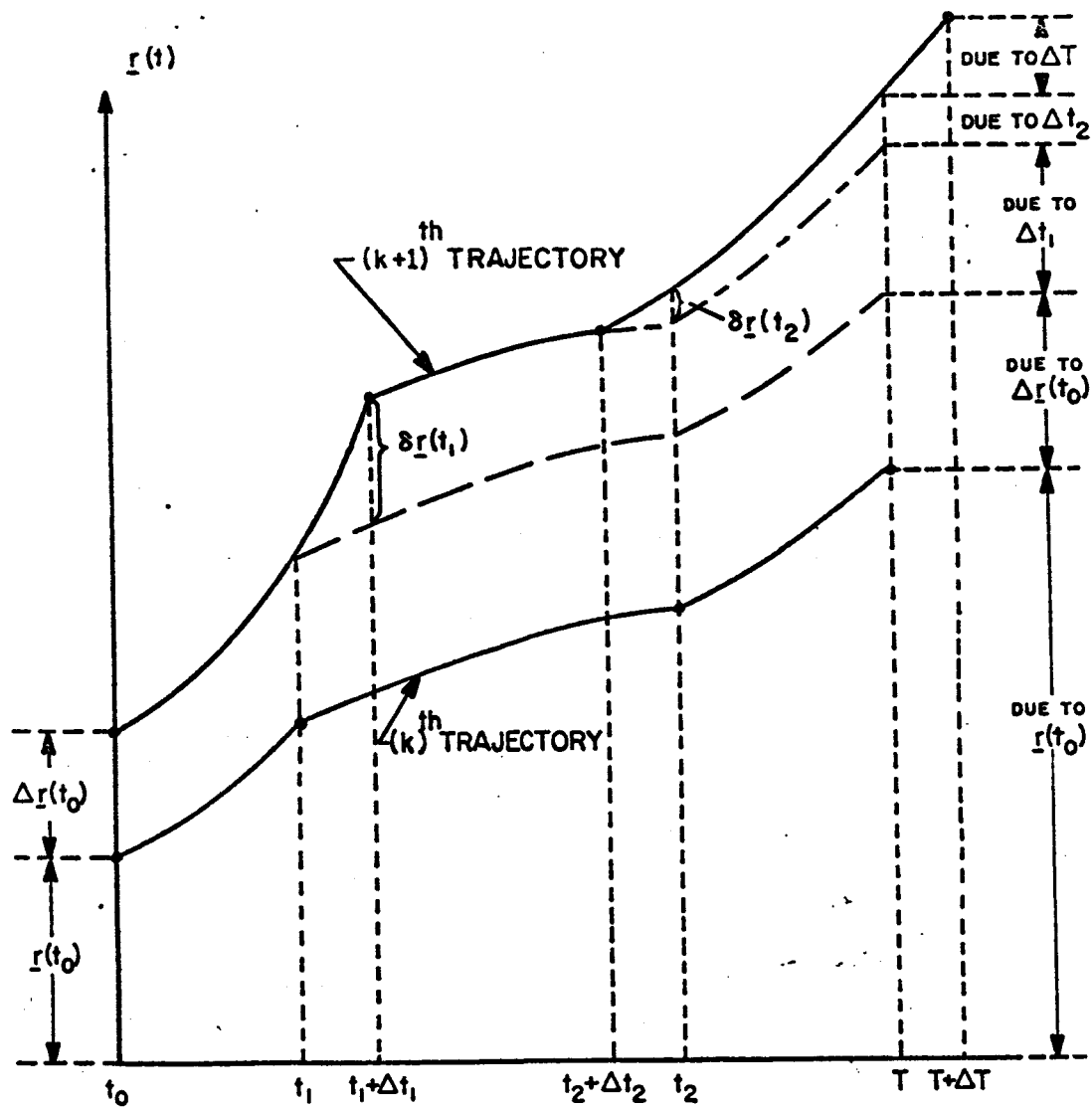


FIGURE 1